Common Calculus Mistakes
Limit Definition of the Derivative

Some problems provide the opportunity for more than one mistake.

The Goal

Use the limit definition of the derivative to find the derivative of

$$f(x) = \sqrt{1-x^2}$$

The Mistakes

Find the mistakes:

1.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2+2xh+h^2}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2}}{\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2}}$$

$$= \lim_{h\to 0}\frac{1-x^2+2xh+h^2-(1-x^2)}{h(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2})} = \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2})}$$

$$= \frac{2x}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$$

Need a hint? Look carefully at the red part:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2\color{red}{+2xh+h^2}}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-x^2\color{red}{+2xh+h^2}}+\sqrt{1-x^2}}{\sqrt{1-x^2\color{red}{+2xh+h^2}}+\sqrt{1-x^2}}$$

$$= \lim_{h\to 0}\frac{1-x^2+2xh+h^2-(1-x^2)}{h(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2})} = \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2})}$$

$$= \frac{2x}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$$

2.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2+2xh+h^2}-\sqrt{1-x^2}}{h}\cdot\left(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2}\right)$$

$$= \lim_{h\to 0}\frac{1-x^2+2xh+h^2+1+x^2}{h} = \lim_{h\to 0}\frac{2-2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}} = 2-2x$$

Need a hint? Look carefully at the red part:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2\color{red}{+2xh+h^2}}-\sqrt{1-x^2}}{h}\cdot\color{red}{\left(\sqrt{1-x^2+2xh+h^2}+\sqrt{1-x^2}\right)}$$

$$= \lim_{h\to 0}\frac{1-x^2+2xh+h^2\color{red}{+1}+x^2}{h} = \lim_{h\to 0}\frac{2-2x\color{red}{\cancel{h}+}h^{\color{red}{\cancel{2}}}}{\color{red}{\cancel{h}}} = 2-2x$$

3.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2-2xh-h^2}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}$$

$$= \lim_{h\to 0}\frac{1-x^2-2xh-h^2-1-x^2}{h\left(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}\right)} = \lim_{h\to 0}\frac{-2x^2-2x\cancel{h}-h^2}{\cancel{h}\left(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}\right)}$$

$$= \frac{-2x^2}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{-2x^2}{2\sqrt{1-x^2}} = \frac{-x^2}{\sqrt{1-x^2}}$$

Need a hint? Look carefully at the red part:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2-2xh-h^2}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}$$

$$= \lim_{h\to 0}\frac{1-x^2-2xh-h^2-1-x^2}{h(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2})} = \lim_{h\to 0}\frac{-2x^2-2x\cancel{h}-h^{\cancel{2}}}{\cancel{h}(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2})}$$

$$= \frac{-2x^2}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{-2x^2}{2\sqrt{1-x^2}} = \frac{-x^2}{\sqrt{1-x^2}}$$

A Correct Solution

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-x^2-2xh-h^2}-\sqrt{1-x^2}}{h}\cdot\frac{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}{\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2}}$$

$$= \lim_{h\to 0}\frac{1-x^2-2xh-h^2-1+x^2}{h(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2})} = \lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}(\sqrt{1-x^2-2xh-h^2}+\sqrt{1-x^2})}$$

$$= \frac{-2x}{\sqrt{1-x^2}+\sqrt{1-x^2}} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

Explanations

All three of the attempted solutions have errors in distributing a negative sign over a factor. The entire expression $(1+x)^2$ has a negative sign in front of it, so when that expression is expanded *every* terms must have its sign changed. The first and second mistakes fail to do this.

The second mistake is compounded with three further errors. First, the limit expression is multiplied by an expression not equal to 1, which therefore changes the value of the expression and the limit. Second, for no apparent reason the sign is reversed on one of the "1"s in the denominator. Third, h is cancelled from the numerator and denominator although h is *not* a *common factor* of the numerator.

In the third mistake the sign error occurs in the step after the rationalization (multiplying the numerator and denominator by the "conjugate expression" of the numerator); the second term is $-(1-x^2) = -1 + x^2$. Once again, h is cancelled from numerator and denominator without being a common factor of the numerator.

Use extreme care with negative signs, especially in problems such as these.