Common Calculus Mistakes
Example: l'Hôpital's Rule

The Goal

Determine

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$$

The Mistake

Find the mistake:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) \overset{\text{l'H}}{=} \lim_{x\to 0}\left(-\frac{1}{x^2}+(\cos(x))^{-2}\right)\text{, does not exist}$$

Need a hint? Look carefully at the red part:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) \overset{\text{l'H}}{=} \lim_{x\to 0}\left(-\frac{1}{x^2}+(\cos(x))^{-2}\right)\text{, does not exist}$$

(The notation "l'H" above the equals sign indicates a step at which l'Hôpital's rule is claimed to be used)

The Correction

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) = \lim_{x\to 0}\left(\frac{\sin(x)-x}{x\sin(x)}\right) \overset{\text{l'H}}{=} \lim_{x\to 0}\left(\frac{\cos(x)-1}{\sin(x)+x\cos(x)}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0}\left(\frac{-\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}\right) = \frac{0}{2} = 0$$

An Explanation

The student has attempted to use l'Hôpital's rule to find the limit of a *difference*. But l'Hôpital's rule is used to find the limit of a *quotient*, when both the numerator and denominator are differentiable on an open interval containing the point of interest (or on an infinite open interval if the limit is to be taken at ±∞) *and* when the limit has indeterminate form, that is, the numerator and denominator both approach zero in the limit or each approaches either ∞ or -∞. Thus the first step is to rearrange the limit expression into the form of an indeterminate quotient.

In this example a correct solution invokes l'Hôpital's rule a second time, since after the first use of the rule, the expression is still indeterminate.